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Fantappié-Arcidiacono Spacetime and Its Consequences in Quantum Cosmology

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Abstract Einstein's gravitational theory gave rise to a new conception of the Universe and Cosmology has been enclosed in the realm of Science and not only of Philosophy as before the Einstein work. Despite this, the presence of the Big Bang singularity, flatness and horizon problems led to the statement that Standard Cosmological Model, based on General Relativity and Standard Model of particle physics, is inadequate to describe the Universe in extreme regimes. Due to this facts, alternative gravitational theories and alternative approaches to cosmology have been proposed during the years. One of the most fruitful approach has been that of Projective Relativity and, in this paper, we analyze the developments of this theory. Projective Relativity, initially proposed by Fantappié and subsequently developed by Arcidiacono, has been recently revisited by prof. Ignazio Licata and other authors. The cosmological consequences of such extension appear relevant. In the following, we analyze the effects of the group approach on the metrics and on the dynamics and we will consider its properties in connection with varying speed of light.

Keywords Cosmology · Relativity · de Sitter

1 Introduction

Einstein's General Relativity can be considered one of the major scientific achievement of last century and its predictions differ significantly from those of Newtonian physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include the gravitational redshift of light, the gravitational time delay, the existence of black holes and of gravitational waves, the phenomenon of gravitational lensing, where multiple images of the same distant astronomical object are visible in the sky, and precession of planetary orbits. For the first time, a comprehensive theory of spacetime, gravity and matter has been formulated giving

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rise to a new conception of the Universe. The current models of cosmology are based on Einstein's equations and if we apply this equations to the whole Universe, we get the relativistic cosmology, in which the cosmological principle is postulated and a model of constant spatial curvature obtained. Predictions, all successful, include the initial abundance of chemical elements formed in a period of primordial nucleosynthesis, the large-scale structure of the Universe and the existence and properties of a thermal echo from the early cosmos, the cosmic background radiation. Despite this, we have to pay close attention to General Relativity, where, inevitably, the application of Einstein's equations to cosmological problems requires an extreme extrapolation of their validity to very far regions of spacetime. Moreover some astronomers, such as Arp and Hoyle, believe that to connect the red shift to the recession is an error because it is known there are other mechanisms which produce the red shift. Arp affirms that some astronomical objects appear to be gravitationally interacting among themselves, and so they should be spatially near. Instead their red shift indicates very different velocities of recession. In addition there are some objects which appear to be older than the Universe and, for this reason, Arp has proposed to return to a variation of the old stationary model in which there is not an origin of time [1–3]. According to this theory, formulated by Bondi, Gold and Hoyle, the Universe has always been as we see it today. The principal problems of the standard relativistic cosmology are the horizon problem and the flatness problem. The standard theory is unable to explain how regions of the Universe that had not been in contact with each other since the Big Bang are observed to emit cosmic background radiation at almost precisely the same temperature as each other and, besides, the relativistic cosmology is unable to provide an explanation as to why the density of the Universe should be so close to the critical value. Due to these facts alternative theories have been considered and one of these approaches are the Extended Theories of Gravity that are obtained by modifying the Einstein-Hilbert action, adding scalar fields or curvature invariants. An original approach to cosmology is that of the Projective Relativity obtained by requiring the laws of physics to be invariant with respect to the Fantappié group, instead of the Poincaré group. Fantappié noted that general relativity follows an approach far from the tradition of mathematical physics in that it does not follow the group structure of physics [4]. Subsequently this theory has been developed by Arcidiacono [5–15] and during the years also other authors have shown interest for this approach to cosmology [16–23]. As it is known, minor changes to the Einstein equations, while exhibiting all the classical verifications, produce completely different cosmologically-interesting solutions and possible universes are numerous. Instead Projective Relativity leads to a univocal Universe and different inertial observers are connected by coordinate transformations whose set is isomorphic to the rotations of the five-dimensional sphere around its centre. The radius of this sphere has no relationship with the distribution of matter or energy over the spacetime. As it is known, Hawking and Hartle have looked for a way, based on quantum mechanics, to explain how time could have spontaneously begun in correspondence with Big-Bang [24]. The idea is that time could have been imaginary, similar to space, near the Big-Bang. That is, in the proximity of the Big-Bang, it would be more exact to speak of 4-dimensional space instead of spacetime. The Hartle-Hawking hypotesis seems to provide a very powerful constraint for the Quantum Cosmology main requirements, but it appears as an ad hoc solution which could be deduced by a fundamental approach [16]. The main difficulty is to understand in what way real time emerges continuously from imaginary time. A possible way-out is the group approach which allows to individuate a Universe model where the entirety of spacetime is represented by a 4-dimensional surface of a 5-dimensional hypersphere, which exists in its entirety and is immutable.

The paper is organized as follows: In Sect. 2 we introduce the Fantappié-Arcidiacono spacetime, while in Sect. 3 we analyze the projective dynamic; Sect. 4 is devoted to the

Hartle-Hawking approach to quantum cosmology; in Sect. 5 we consider the variation of speed of light and its consequences in atomic models while in Sect. 6 we analyze the Projective General Relativity; Finally in Sect. 7 we give the conclusion.

2 Fantappié-Arcidiacono Spacetime

The problem to improve the Einstein gravitational theory, can be faced remaining in four dimensions or considering manifolds with more than four dimensions. The most interesting among all the 4-dimensional unitary theories is that of Einstein [25]. In this theory the metric tensor is decomposed into its symmetric and antisymmetric parts and accordingly we also have nonsymmetric Christoffel's symbols. He, in this way, succeeded in unifying, mathematically, the gravitational and electromagnetic field. The Einstein approach in a first moment aroused great interest but then it resulted of difficult physical interpretation. In 1921 Theodor Kaluza extended general relativity to a five-dimensional spacetime introducing the following metric [26]

$$\begin{aligned} ds^2 = \gamma_{AB} dx^A dx^B = \gamma_{ik} dx^i dx^k + 2\gamma_{i5} dx^i dx^5 + \gamma_{55}(dx^5)^2, \\ i, k = 1, \ldots, 4, \ A, B = 1, \ldots, 5. \end{aligned} \tag{1}$$

In this way it is possible to write the 5-dimensional Einstein equations

$$G_{AB} = \chi T_{AB} \tag{2}$$

and, to write the equations in the 4-dimensional spacetime, he set

$$\gamma_{55} = 1, \quad \gamma_{i5} = \lambda A_i \tag{3}$$

where λ is an universal constant and we have

$$g_{ik} = \gamma_{ik} - \gamma_{i5}\gamma_{k5} = \gamma_{ik} - \lambda^2 A_i A_k \tag{4}$$

where g_{ik} is the 4-dimensional metric tensor.

By using this metric, the Einstein equations can be separated out into further sets of equations, one of which is equivalent to Einstein field equations, another set equivalent to Maxwell's equations for the electromagnetic field. In 1926, Oskar Klein proposed that the fourth spatial dimension is curled up in a circle of very small radius, so that a particle moving a short distance along that axis would return to where it began. This means the fourth dimension would have the topology of a circle, with a radius of the order of the Planck length [27]. Five-dimensional spacetime then has topology $R^4 \times S^1$and the fifth coordinate y is periodic, $0 \le my \le 2\pi$, where m is the inverse radius of the circle. In our normal perception of spacetime we would never be able to see this extra dimension. Because of the periodicity of the extra dimension we can make a Fourier expansion in this coordinate and we would end up with an infinite tower of fields in four dimensions. Kaluza-Klein theory can be extended to cover the other fundamental forces, the weak and strong forces, and this approach to unification of forces is taken by some more modern theories as string theory and the related M-theory.

An original approach to spacetime is El Naschie's E-Infinity theory which regards discontinuities of space and time in a transfinite way [28–30]. Introducing a new Cantorian spacetime, El-Naschie admitted formally an infinite dimensional real spacetime, which is hierarchical in a strict mathematical way [31–34]. By considering the classical triadic Cantor set in n-dimensional space, and writing the Hausdorff dimensions using the bijection formula $d_c^{(n)} = (1/d_c^{(0)})^{n-1}$ we find [31]

$$\begin{aligned}
d_c^{(0)} &= \ln 2/\ln 3 \cong 0.630929753\\
d_c^{(1)} &= 1\\
d_c^{(2)} &= \ln 3/\ln 2 \cong 1.584962502\\
d_c^{(3)} &\cong 2.512106129\\
d_c^{(4)} &\cong 3.981594012\\
d_c^{(5)} &\cong 6.3106770202\\
d_c^{(6)} &\cong 10.00218672 \cong 10\\
&\vdots
\end{aligned}$$

We have that at low resolution or equivalently at low energy $d_c^{(n)} \cong n$ only when $n = 4$. Thus spacetime has Cantorian structure at the small scale, or equivalently at high energy resolution and in the case of random Cantor set $d_c^{(0)} = \phi = (\sqrt{5} - 1)/2$ and we get $d_c^{(4)} = 4 + \phi^3 = 4.236067977$. In conclusion we can say that the Einstein's four-dimensional spacetime may be considered to be only an approximation valid only for the large structure of spacetime. Symmetry is one of the most fundamental properties of nature and the branch of mathematics dealing with symmetry is the group theory. The group theory is extremely important and plays a fundamental role in particle physics. Lie groups lie at the intersection of two fundamental fields of mathematics: algebra and geometry. A Lie group is first of all a group and secondly it is a differentiable manifold. Therefore a Lie group is a group which is also a differentiable manifold, with the property that the group operations are compatible with the differential structure. Cartan has given a complete classification of the simple groups and he has found that there are four infinite families:

1) A_n groups; a model of these groups is $SU(n+1)$
2) B_n groups; a model of these groups is $SO(2n+1)$
3) C_n groups; a model of these groups is $Sp(2n)$
4) D_n groups; a model of these groups is $SO(2n)$.

$Sp(2n)$ denotes the symplectic group. Then there are other five possible simple groups and that is the so-called exceptional cases $G2$, $F4$, $E6$, $E7$, and $E8$. These cases are said exceptional because they do not fall into infinite series of groups of increasing dimension. $G2$ has 14 dimensions, $F4$ has 52 dimensions, $E6$ has 78 dimensions, $E7$ has 133 dimensions and $E8$ has 248 dimensions. The $E8$ algebra is the largest and most complicated of these exceptional cases, and is often the last case of various theorems to be proved. Mathematicians have mapped the inner structure of E_8 and these developments are very important because there are many connections between E_8 and other areas as, for example, string theory. E_8 is a very beautiful group in fact it describes the symmetries of a particular 57-dimensional object while E_8 itself is 248-dimensional. E_8 is completely abstract and some authors link it to the hierarchy of E-Infinity theory [35–37]. Fantappié has used group theory for studying cosmology obtaining as symmetry group $S(O_5)$ and that is the group of rotation of the Euclidean 5-dimensional space. As reported in [6, 16] Fantappié's starting point was the study of classical and relativistic physics spacetime. Spacetime of classical physics is a 4-dimensional manifold endowed with the following geometrical structures:

1) Absolute time;
2) Absolute space;
3) Absolute spatial distances;
4) Absolute temporal distances.

Space-time of special relativity, instead, is a 4-dimensional manifold endowed with the following geometrical structures:

1) Relative time;
2) Relative space;
3) Absolute spacetime distances.

Galileo‘s group has order 10 and expresses Galileo’s well-known relativity principle. Moving on to relativistic physics, spatial rotations and inertial movements are blended in a unique operation, the rotations of an Euclidian space M_4, characterized by 6 parameters,

$$x_i' = a_{ik} x_k, \tag{5}$$

where $|a_{ik}| = 1$, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$.

These transformations, called Lorentz’s special transformations, form Lorentz’s proper group and joining the reflections, form Lorentz’s extended group. Then we need to add the translations of M_4

$$x_i' = x_i + a_i, \tag{6}$$

characterized by 4 parameters, which comprise spatial and temporal translations. By composing the transformations of these two groups, we obtain Lorentz’s general transformations which form Poincaré’s group of 10 parameters

$$x_i' = a_{ik} x_k + a_i. \tag{7}$$

Poincaré’s group mathematically translates Einstein’s relativity principle. When $c \to \infty$ so that $\frac{v}{c} \ll 1$, Minkowski’s spacetime reduces to that of Newton’s and Poincaré’s group reduces to Galileo’s group.

Fantappié went on this direction and tried to understand if Poincaré’s group could be the limit of a more general group, in the same manner as Galileo’s group is the limit of Poincaré’s group. In [4] he wrote a new group of transformations which had as limit Poincaré’s group and he was able also to demonstrate that his group was not able to be the limit of any continuous group of 10 parameters. That is, by limiting to groups of 10 parameters and to 4-dimensional spaces, what happened with Galileo’s and Poincaré’s groups cannot be repeated. For this reason this group is called the final group.

Fantappié’s group is characterized by two constants: speed of light c and a radius of spacetime r. This group determines an Universe endowed with a perfect symmetry: De Sitter’s Universe. Let us remember that De Sitter’s Universe is obtained from Einstein’s equations with a positive cosmological constant and the solution of motion equation is the following

$$r(t) = r(0) e^{ct\sqrt{\Lambda/3}}.$$

This model is a 4-dimensional hyperboloid in real time and a 4-dimensional sphere in imaginary time.

Moving from De Sitter spacetime, Arcidiacono showed that, through a flat projective representation, one could obtain a spacetime which generalizes Minkowski's spacetime. Let us consider the homogeneous coordinates defined as

$$x_k = r\overline{x}_k/\overline{x}_5, \tag{8}$$

then these transformations form the group of 5-dimensional rotations. We can classify them in three categories as follows [6]:

a) *Time translations*: considering two observers still standing in the same place, but separated by a great distance in time, that is, the same observer in two different moments

$$\begin{cases} x'_\alpha = \dfrac{x\sqrt{1-\eta^2}}{1-\eta t/(r/c)}, \\ t' = \dfrac{t-T_0}{1-\eta t/(r/c)}, \end{cases} \tag{9}$$

with $\eta = \frac{T_0}{r/c} = \frac{T_0}{t_0}$ where T_0 is the parameter of time translation. It follows that for $t = \pm r/c$, one has $x' = 0$.

These transformations for $r \to \infty$ are reduced to the classic time translations

$$x' = x, \qquad t' = t - T_0. \tag{10}$$

b) *Spatial translations*: considering two observers at the same time and still standing compared to each other, but separated by great distance in space (for example along the x-axis)

$$\begin{aligned} x' &= \frac{x-S}{1+\alpha x/r}, \\ y' &= \frac{y\sqrt{1+\alpha^2}}{1+\alpha x/r}, \\ z' &= \frac{z\sqrt{1+\alpha^2}}{1+\alpha x/r}, \\ t' &= \frac{t\sqrt{1+\alpha^2}}{1+\alpha x/r}, \end{aligned} \tag{11}$$

with $\alpha = \frac{S}{r}$ and where S is the parameter of translation along the x-axis.

At the relativistic limit, that is for $r \to \infty$, (11) reduce themselves to

$$x' = x - S, \qquad y' = y, \qquad z' = z, \qquad t' = t. \tag{12}$$

c) *Pullings*: considering two observers that initially coincide, and one moving rectilinearly and uniformly to the other, with velocity parallel to the x-axis

$$\begin{cases} x' = \dfrac{x - Vt}{\sqrt{1-\beta^2}}, \\ y' = y, \\ z' = z, \\ t' = \dfrac{t - Vx/c^2}{\sqrt{1-\beta^2}}. \end{cases} \tag{13}$$

The projective spacetime coordinates are regulated by Fantappié's transformations and the relations that link projective and non-projective coordinates are the following

$$\begin{cases} T = T_b \operatorname{tgh} \dfrac{t}{T_b} \\ X = r \operatorname{tg} \dfrac{x}{r} \end{cases}, \qquad \begin{cases} t = \dfrac{T_b}{2} \log \dfrac{T_b + T}{T_b - T} \\ x = r \operatorname{arctg} \dfrac{X}{r} \end{cases}, \tag{14}$$

with $T_b \cong 15 \times 10^9$ year and $r \cong 142000 \times 10^{18}$ km.

This physical interpretation of Fantappié's transformations implies that the projective age of the Universe is constant. The temporal translations demonstrate

$$T = \frac{T_1 + T_2}{1 + T_1 T_2 / T_b^2}. \tag{15}$$

This relation is equal in form to relativistic law of the composition of velocities. Therefore, as such, the speed of light is the same for each observer in whichever motion, and as being finite, cannot be exceeded.

Let us recall that relativistic metric is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{16}$$

with

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{17}$$

By following Fantappié transformations, instead, the metric of spacetime can be written as

$$L^2 ds^2 = L\left(\sum_{i=1}^{4} dx_i dx_i\right) - \left(\sum_{i=1}^{4} \frac{x_i}{r} dx_i\right)^2 \tag{18}$$

with

$$L = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - \frac{c^2}{r^2} t^2 + 1 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - \left(\frac{t}{t_b}\right)^2 + 1 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - \eta^2 + 1. \tag{19}$$

At the relativistic limit, that is for $r \to \infty$, this metric is reduced to Minkowski's metrics in fact

$$\begin{cases} L \to 1, \\ \sum \frac{x_i}{r} dx_i \to 0. \end{cases} \tag{20}$$

In the Fantappié-Arcidiacono Universe, therefore, the metrics reads

$$ds^2 = f_{\mu\nu} dx_\mu dx_\nu \tag{21}$$

with

$$f_{\mu\nu} = \begin{pmatrix} \frac{1}{L} - \frac{x_1^2}{L^2r^2} & -\frac{x_1x_2}{L^2r^2} & -\frac{x_1x_3}{L^2r^2} & -\frac{x_1x_4}{L^2r^2} \\ -\frac{x_1x_2}{L^2r^2} & \frac{1}{L} - \frac{x_2^2}{L^2r^2} & -\frac{x_2x_3}{L^2r^2} & -\frac{x_2x_4}{L^2r^2} \\ -\frac{x_1x_3}{L^2r^2} & -\frac{x_2x_3}{L^2r^2} & \frac{1}{L} - \frac{x_3^2}{L^2r^2} & -\frac{x_3x_4}{L^2r^2} \\ -\frac{x_1x_4}{L^2r^2} & -\frac{x_2x_4}{L^2r^2} & -\frac{x_3x_4}{L^2r^2} & \frac{1}{L} - \frac{x_4^2}{L^2r^2} \end{pmatrix}. \tag{22}$$

Therefore, the metric is function of spacetime also without gravitational field.

In the special relativity, we have

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = -c^2d\tau^2. \tag{23}$$

Therefore

$$c^2d\tau^2 = c^2dt^2\left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\right) = c^2dt^2\left(1 - \frac{v^2}{c^2}\right) \tag{24}$$

and so

$$d\tau = dt\sqrt{1 - \beta^2}. \tag{25}$$

In the projective Universe instead

$$\begin{aligned} L^2ds^2 &= L(dx^2 + dy^2 + dz^2 - c^2dt^2) - \left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz - \frac{c^2t}{r}dt\right)^2 \\ &= L(dx^2 + dy^2 + dz^2 - c^2dt^2) - \left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz - \eta c dt\right)^2 \\ &= Lc^2dt^2(\beta^2 - 1) - \left[cdt\left(\frac{xdx}{rcdt} + \frac{ydy}{rcdt} + \frac{zdz}{rcdt} - \eta\right)\right]^2 \\ &= Lc^2dt^2(\beta^2 - 1) - c^2dt^2\left(\frac{x}{r}\frac{v_x}{c} + \frac{y}{r}\frac{v_y}{c} + \frac{z}{r}\frac{v_z}{c} - \eta\right)^2 \\ &= c^2dt^2\left[L(\beta^2 - 1) - \left(\frac{x}{r}\frac{v_x}{c} + \frac{y}{r}\frac{v_y}{c} + \frac{z}{r}\frac{v_z}{c} - \eta\right)^2\right]. \end{aligned}$$

Therefore we can write

$$ds^2 = \frac{c^2dt^2[L(\beta^2 - 1) - (\frac{x}{r}\frac{v_x}{c} + \frac{y}{r}\frac{v_y}{c} + \frac{z}{r}\frac{v_z}{c} - \eta)^2]}{L^2} = -c^2d\tau^2.$$

Then we obtain

$$d\tau = \frac{dt\sqrt{L(1-\beta^2) + (\frac{x}{r}\frac{v_x}{c} + \frac{y}{r}\frac{v_y}{c} + \frac{z}{r}\frac{v_z}{c} - \eta)^2}}{L} = dt\frac{M}{L}. \tag{26}$$

In relativistic spacetime, the mass is only function of velocity, in fact

$$m = m_0\frac{dt}{d\tau} = \frac{m_0}{\sqrt{1-\beta^2}}. \tag{27}$$

In projective spacetime instead

$$m = m_0\frac{dt}{d\tau} = m_0\frac{L}{M}. \tag{28}$$

Therefore the mass is also function of spacetime coordinates. We study the following three interesting cases:

1) In the system of the observer for $t = 0$ we have

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$

in agreement with special relativity

2) For the masses that move following Hubble's law we obtain

$$m = m_0\frac{\frac{x^2}{r^2}+1}{\sqrt{(\frac{x^2}{r^2}+1)(1-\frac{H^2x^2}{c^2}) + \frac{x^2}{r^2}\frac{H^2x^2}{c^2}}}$$

$$m_0\frac{\frac{x^2}{r^2}+1}{\sqrt{(\frac{x^2}{r^2}+1)(1-\frac{x^2}{r^2}) + \frac{x^4}{r^4}}} = m_0\left(\frac{x^2}{r^2}+1\right).$$

The mass increases with the distance.

3) For $x = \beta = 0$ we have

$$m = m_0(1-\eta^2).$$

At the time of Big Bang all the masses tend to zero.

3 Projective Dynamics

In [21] we considered the dynamical properties of projective spacetime and we shortly resume the gotten results. The projective 4-velocity is defined by the following relation

$$\begin{cases} U_\alpha = v_\alpha\dfrac{L}{M}, \\ U_4 = c\dfrac{L}{M}. \end{cases} \tag{29}$$

For $r \to \infty$ we have

$$\begin{cases} L \to 1 \\ M \to \sqrt{1-\beta^2} \end{cases} \tag{30}$$

then we obtain the relativistic 4-velocity. Fantappié's transformations represent 5-dimensional rotations, [6], and therefore we defined the following 5-vectors:

$$\begin{cases} \mathbf{U}_\mu = \dfrac{d\overline{x}_\mu}{d\tau} \\ \mathbf{P}_\mu = m_0 \mathbf{U}_\mu \end{cases} \qquad (\mu = 1, 2, 3, 4, 5) \tag{31}$$

Remembering that

$$x_\mu = r\frac{\overline{x}_\mu}{\overline{x}_5}$$

we obtain

$$U_\mu = \frac{dx_\mu}{d\tau} = \frac{r(\mathbf{U}_\mu \overline{x}_5 - \mathbf{U}_5 \overline{x}_\mu)}{\overline{x}_5^2}. \tag{32}$$

By multiplying for m_0, it follows that

$$P_\mu = \frac{r(\mathbf{P}_\mu \overline{x}_5 - \mathbf{P}_5 \overline{x}_\mu)}{\overline{x}_5^2}. \tag{33}$$

These are the relations that link 4-dimensional and 5-dimensional vectors. The following equation

$$f = \frac{d}{dt}\left(m_0 v \frac{L}{M}\right) \tag{34}$$

represents the classical 3-dimensional force, where t is the projective time, and now we introduce 5-dimensional force defined as

$$\mathbf{F} d\tau = d\mathbf{P}. \tag{35}$$

By replacing the definition of the proper time we can write

$$\mathbf{F} dt \frac{M}{L} = d\mathbf{P} \tag{36}$$

For small variations of $\overline{x}_5$ it is possible to write

$$\begin{cases} \mathbf{P}_1 = P_1 \dfrac{\overline{x}_5}{r} \\ \mathbf{P}_2 = P_2 \dfrac{\overline{x}_5}{r} \\ \mathbf{P}_3 = P_3 \dfrac{\overline{x}_5}{r} \\ \mathbf{P}_4 = P_4 \dfrac{\overline{x}_5}{r} \\ \mathbf{P}_5 = 0 \end{cases} \tag{37}$$

By dividing spatial component by temporal component we obtain

$$\begin{cases} \dfrac{\mathbf{P}_1}{\mathbf{P}_4} = \dfrac{P_1 \frac{\overline{x}_5}{r}}{P_4 \frac{\overline{x}_5}{r}} = \dfrac{m_0 L v_1}{M} \dfrac{M}{m_0 L c} = \dfrac{v_1}{c} \\ \dfrac{\mathbf{P}_2}{\mathbf{P}_4} = \dfrac{v_2}{c} \\ \dfrac{\mathbf{P}_3}{\mathbf{P}_4} = \dfrac{v_3}{c} \end{cases} \tag{38}$$

It follows that

$$\frac{\widehat{P}}{\mathbf{P}_4} = \frac{v}{c} \quad \Rightarrow \quad v = \frac{\widehat{P} c}{\mathbf{P}_4}. \tag{39}$$

Considering that, in the same approximation,

$$\mathbf{P}^2 = \left|\widehat{P}\right|^2 - \mathbf{P}_4^2 = -m_0^2 c^2 \tag{40}$$

we obtain

$$\widehat{P} \cdot d\widehat{P} - \mathbf{P}_4 d\mathbf{P}_4 = 0. \tag{41}$$

Consequently,

$$\frac{v}{c} \cdot d\widehat{P} = d\mathbf{P}_4 \quad \Rightarrow \quad f \cdot v dt \frac{\overline{x}_5}{r} = c d\mathbf{P}_4. \tag{42}$$

Therefore the previous equation becomes

$$dE = \frac{r}{\overline{x}_5} c d\mathbf{P}_4. \tag{43}$$

Then we have the relation

$$E = \mathbf{P}_4 c \frac{r}{\overline{x}_5} + A = m_0 c^2 \frac{L}{M} + A \tag{44}$$

where A is an integration constant. Now we consider a particle initially at rest, and then we apply a force on it. We can write

$$\frac{\overline{x}_5}{r}\int_0^t f\cdot v dt = [c\mathbf{P}_4]_0^t \tag{45}$$

From (44), (26) we have

$$L = E - E_0 = m_0 c^2 \frac{L}{M} - m_0 c^2 \frac{L}{\sqrt{\frac{x^2}{r^2}+1}}. \tag{46}$$

For example let us write the following system

$$\begin{cases} \frac{d}{dt}\left(m_0 v \frac{L}{M}\right) = f \\ x(0) = 0 \\ v(0) = 0 \end{cases} \tag{47}$$

where f is a constant force. Then, we obtain

$$d\left(m_0 v \frac{L}{M}\right) = f dt \quad \Rightarrow \quad m_0 v = f t \frac{M}{L}. \tag{48}$$

In conclusion

$$v = at\frac{M}{L}. \tag{49}$$

In the relativistic limit this relation can be written

$$v = at\sqrt{1-\beta^2} = \frac{at}{\sqrt{1+(at/c)^2}}. \tag{50}$$

This is the expression of the uniformly accelerated motion in special relativity.

4 Hartle-Hawking Quantum Cosmology

We emphasize here that the cosmological inflationary models if they solve the flatness and horizon problems, do not solve the initial singularity problem. In the usual approach to quantum gravity the Universe is assumed to be a quantum system [38–42] or as a classical background with primordial quantum processes, as in the context of quantum field theory on curved spacetime [43]. This approach is based on the Wheeler-De Witt equation

$$\left(G_{ijkl}\frac{\partial^2}{\partial h_{ij}\partial h_{kl}} - {}^{(3)}R h^{1/2} + 2\Lambda h^{1/2}\right)\psi(h_{ij}) = 0, \tag{51}$$

where h_{ij} is the spatial metric, ${}^{(3)}R$ is the scalar curvature of the intrinsic geometry of the three-surface and

$$G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}). \tag{52}$$

Let us remember that, in the construction of cosmological models, it is not possible to deduce the contour conditions around the outside of the Universe. We can choose many different conditions, but we need to calculate their consequences to see if they agree with the observations. The idea of Hartle and Hawking is that time could have been imaginary near Big Bang and they eliminated the problem of contour conditions because their Universe has no frontier. Expanding this idea to the whole spacetime manifold, we find ourselves in a model of the Universe in which geometry is linked to the group that Fantappié obtained by generalizing Poincaré's relativistic group. The entirety of spacetime is represented by a 4-dimensional surface of a 5-dimensional hypersphere, which exists in its entirety and is immutable and all events, past, present and future, simply exist in the Universe of imaginary time. By sectioning the sphere with planes orthogonal to the coordinate lines of imaginary time, one sees that this model represents a stationary Universe of cyclic imaginary time. By transforming imaginary time into real time, we obtain the passage from a spacetime hypersphere of imaginary time, to a hyperboloid of real time. Perhaps the imaginary time is the fundamental structure of the Universe, while real time simply originates from our senses. One of the most predictions of General Relativity is the existence of horizons in many of its solutions. The most popular solution of the Einstein's equations is the Schwarzschild one, which represent the external metric of black-holes.

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{1 - 2GM/c^2 r} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{53}$$

This line element has a singularity at

$$r_s = \frac{2GM}{c^2} \tag{54}$$

namely the Schwarzschild radius. The Schwarzschild metric can be generalized to the Kerr-Newman solution which describes the most general stationary black hole exterior and the only parameters appearing in this solution are the mass M, the angular momentum J and the electric charge Q of the Black Hole. This result is known in the literature as the no-hair theorem. Hawking has shown that the black hole radiates and this radiation is thermal corresponding to the following temperature:

$$\theta_b = \frac{hc^3}{16\pi^2 GMk} \tag{55}$$

The solution in real time of Einstein's equations with positive cosmological constant in absence of energy and matter is the De Sitter space. This can be included as a hyperboloid in the 5-dimensional Minkowski space with metric

$$ds^2 = -dt^2 + \frac{1}{H^2}\cosh Ht[dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{56}$$

Setting $\tau = it$ we obtain the Euclidian metric in imaginary time

$$ds^2 = -d\tau^2 + \frac{1}{H^2}\cos H\tau[dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{57}$$

Hawking has shown that the De Sitter space has thermal property similar to the black hole. He writes the De Sitter metric in a static form [24]

$$ds^2 = -(1 - H^2r^2)dt^2 + (1 - H^2r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{58}$$

There is a singularity in $r = \frac{1}{H}$, in analogy with the Schwarzschild solution's case, which can be eliminated by a change of coordinates. The coordinate of the imaginary time is periodic with period $\beta = \frac{2\pi}{H}$ and for the observers in De Sitter Universe it implies the possibility to define a temperature, an entropy and an area of the horizon, respectively given by [16]

$$\theta_D = \frac{H}{2\pi} = \frac{1}{\beta}; \qquad S = \frac{\pi}{H^2} = \frac{\beta^2}{4\pi}; \qquad A = \frac{4\pi}{H^2} = \frac{\beta^2}{\pi}. \tag{59}$$

From previous relations we see that

$$S = \frac{1}{4}A \tag{60}$$

which is the expression of the t'Hooft-Susskind-Bekenstein Holographic Principle. We can summarize by saying that Hawking has shown that De Sitter space, and all the quantum fields in it, behave as if they were at a temperature $\frac{H}{2\pi}$, and we could observe the consequences of it in fluctuations of the cosmic radiation. A problem arises since the De Sitter space, as a solution of the Einstein's equations, is empty and there is only the cosmological constant in disagreement with what we see. Instead, through the group theoretical approach, we get it without using the general relativity but simply as a generalization of Poincaré's group. Therefore as commented by prof. Licata [16, 17], the Fantappié-Arcidiacono approach seems to be an excellent starting point to resolve the problem of the foundation of quantum cosmology.

5 Varying Speed of Light

Recently, many authors have proposed cosmological models where a variation with cosmological time of the speed of light is hypothesized as a viable alternative to inflation, in order to solve the classical problems of cosmology [44–47]. For example, in [44] the authors consider the cosmological implications of light travelling faster in the early Universe than today and they propose a prescription for deriving corrections to the cosmological evolution equations while the speed of light is changing. In this VSL (Varying Speed of Light) theory two scenarios were considered; in [44] the speed of light varies abruptly at a critical temperature, while in [45] the author considered scenarios in which the speed of light varies like a power of the scale factor. In the VSL model, the authors write the Friedmann relations in the presence of a time-dependent speed of light

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2(t)}\right) + \frac{\Lambda c^2(t)}{3}, \tag{61}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2(t)}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2(t)}{3}, \tag{62}$$

where $a(t)$ is the scale factor, p is the fluid pressure, ρ is the fluid density, k is the curvature parameter, Λ is the cosmological constant and all derivatives are with respect to comoving proper time. From these relations they derive the modified matter conservation

$$\rho + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2(t)}\right) = \frac{3kc^2(t)\dot{c}(t)}{4\pi G a^2}. \tag{63}$$

If we have a time variation of Newton's constant we get

$$\rho + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2(t)}\right) = -\rho\frac{\dot{G}}{G} + \frac{3kc^2(t)\dot{c}(t)}{4\pi G a^2}. \tag{64}$$

In [44] $\hbar \propto c$ and the quantity $\hbar/c$ is constant. The symbol $\propto$ is taken to mean 'proportional to' throughout this paper. They assume that the mass and the electric charge are constants and therefore the fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \propto \frac{1}{c^2}$. In [45] Barrow assumes that the speed of light varies as some power of the expansion scale factor

$$c(t) = c_0 a^n \tag{65}$$

and shows that the flatness and horizon problems can be solved if $n \leq -1$ in a radiation dominated Universe or $n \leq -1/2$ in a matter dominated Universe. In the group approach, instead, the speed of light varies according to a relation deduced by the Fantappié transformations. In fact, in the Fantappié-Arcidiacono projective relativity, if a light ray moves in the absolute spacetime, then the speed of light is c_0 and to know the corresponding value in the projective spacetime it is necessary to evaluate

$$c = \frac{dX}{dT} = \frac{dX}{dx}\frac{dx}{dt}\frac{dt}{dT} = \frac{c_0}{(1-\eta^2)\cos^2(\frac{x}{R})} = \frac{c_0}{(1-\eta^2)\cos^2(\text{arctg}\frac{X}{R})}.$$

Therefore we have $c = c_0$ only when the spatial and temporal distances are not comparable with the Universe radius. The speed of light is a function of spacetime length and, once defined a point in the space, it increases by going back in time. The relation that ties the speed of light to the time is the following

$$c = \frac{c_0}{1-\eta^2}. \tag{66}$$

Let us suppose to have two inertial frames of reference A and B with a relative speed of leaving equal to v (in the De Sitter spacetime). From the viewpoint of A (in the projective spacetime), when the distances grow bigger, the frame of reference B seems to increase the leaving speed and that speed increases of the factor

$$\frac{1}{(1-\eta^2)\cos^2(\text{arctg}\,\frac{X}{R})}.$$

From the viewpoint of B (in the projective spacetime), it is the frame of reference A to accelerate, at first slowly, then faster and faster. Therefore, in the projective spacetime, a body not subjected to any force or follows the Universe expansion or moves with an accelerated motion while in the De Sitter space it moves with constant speed. In other words, an inertial motion in the De Sitter space can appear as accelerated motion in the projective space. Therefore if a galaxy in addition to the expanding motion has also a further own speed of leaving v we measure in the projective space

$$V = HX + \frac{v}{(1-\eta^2)\cos^2(\text{arctg}\,\frac{X}{R})}.$$

If $(1-\eta^2)\cos^2(\text{arctg}\frac{X}{R}) \ll 1$ then the galaxy seems not to obey Hubble's law and Universe seems accelerated or decelerated. In particular, it is accelerated if its relative speed with respect to us is in opposite direction; otherwise it will appear decelerated.

6 Projective General Relativity

Let us remember that, following the definition of Cartan, any Riemann manifold is associated with an infinite family of Euclidean spaces tangent to it in each of its P points. These infinity spaces are joined by a connection law and are individuated by a holonomy group. By introducing a local coordinates system y^i and a linear forms, ω^i, of differential dy^i we can write $ds^2 = \omega_s \omega^s$. If we consider, on the tangent space to a point P, four orthogonal vectors e_i we have [8]

$$\begin{cases} dP = \omega^i e_i \\ de_i = \omega_i^k e_k \\ e_i e_k = \delta_{ik} \end{cases} \tag{67}$$

where $\omega_k^i = \gamma_{ks}^i \omega^s$ and γ_{ks}^i are the Ricci rotation coefficients. If the point P and the associated reference frame describe a closed infinitesimal cycle on the tangent space, in general the vector e_i' doesn't coincide with e_i and the cycle is open. It can be closed through a translation Ω^i and a rotation Ω_k^i on the tangent space and we have

$$\begin{cases} \Omega^i = d\omega^i + \omega_s^i \wedge \omega^s \\ \Omega_k^i = d\omega_k^i + \omega_s^i \wedge \omega_k^s \end{cases} \tag{68}$$

where Ω^i is the torsion and Ω_k^i is the curvature. To develop the projective general relativity we have to introduce a 5-dimensional Riemann manifold which allows as holonomy group the Fantappié one, isomorphic to the 5-dimensional rotations group, and the gravitation equations are

$$R_{AB} - \frac{1}{2} R g_{AB} = \chi T_{AB} \quad (A, B = 0, 1, 2, 3, 4) \tag{69}$$

where g_{AB} are the coefficients of the five-dimensional metric. We immediately understand, from how much we have said above, that, in projective general relativity, is fundamental the geometry of projective connections and therefore let us remember the following concepts.

If we have a differentiable manifold M and it symmetric connection θ, the curvature tensor in local coordinate is

$$K_{\beta\gamma\delta}^{\alpha} = \frac{\partial \theta_{\beta\delta}^{\alpha}}{\partial x^{\gamma}} - \frac{\partial \theta_{\beta\gamma}^{\alpha}}{\partial x^{\delta}} + \sum_{\sigma} (\theta_{\sigma\gamma}^{\alpha} \theta_{\beta\delta}^{\sigma} - \theta_{\sigma\delta}^{\alpha} \theta_{\beta\gamma}^{\sigma}). \tag{70}$$

The Ricci tensor of connection is

$$K_{\beta\delta} = \sum_{\alpha} K_{\beta\alpha\delta}^{\alpha}. \tag{71}$$

By setting

$$A_{\beta\delta} = \frac{1}{2}(K_{\beta\delta} - K_{\delta\beta}) = \frac{1}{2} \sum_{\alpha} \left(\frac{\partial \theta_{\alpha\delta}^{\alpha}}{\partial x^{\beta}} - \frac{\partial \theta_{\alpha\beta}^{\alpha}}{\partial x^{\delta}} \right) \tag{72}$$

the following tensor

$$W_{\beta\gamma\delta}^{\alpha} = K_{\beta\gamma\delta}^{\alpha} - \frac{2}{n+1} \delta_{\beta}^{\alpha} A_{\gamma\delta} - \frac{1}{n-1} (\delta_{\gamma}^{\alpha} K_{\beta\delta} - \delta_{\delta}^{\alpha} K_{\beta\gamma})$$

$$+\frac{2}{n^2-1}(\delta^\alpha_\gamma A_{\beta\delta}-\delta^\alpha_\delta A_{\beta\gamma}) \tag{73}$$

is said projective curvature tensor. Two such connections are projectively equivalent if they define the same geodesics up to parametrization.

A n-dimensional differentiable manifold M with symmetric connection θ is said locally projectively flat if $\forall x \in M$ there is a neighborhood U and a diffeomorphism from U to an open of $\mathbb{R}^n$ which transforms the images of geodesics of θ, contained in U, into straight lines. Let us remember that M is locally projectively flat if and only if the projective curvature tensor is identically zero and, if the Ricci tensor is symmetric, the manifold M is locally projectively flat if and only if the curvature tensor can be written with the following relation

$$K^\alpha_{\beta\gamma\delta}=\frac{1}{n-1}(\delta^\alpha_\gamma K_{\beta\delta}-\delta^\alpha_\delta K_{\beta\gamma}). \tag{74}$$

Given a Riemannian manifold M and u and v, two linearly independent tangent vectors at the same point x_0, we can define

$$K(u,v)=\left[\frac{\sum R_{\alpha\beta\gamma\delta}u^\alpha v^\beta u^\gamma v^\delta}{\sum(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})u^\alpha v^\beta u^\gamma v^\delta}\right](x_0). \tag{75}$$

It can be shown that $K(u,v)$ depends only on the plane spanned by u and v and it is called sectional curvature. In a Riemannian manifold the relation (74) can be written

$$R^\alpha_{\beta\gamma\delta}=\frac{1}{n-1}(\delta^\alpha_\gamma R_{\beta\delta}-\delta^\alpha_\delta R_{\beta\gamma}) \tag{76}$$

and setting as usual

$$R_{\alpha\beta\gamma\delta}=\sum_\rho g_{\alpha\rho}R^\rho_{\beta\gamma\delta} \tag{77}$$

we can write

$$R_{\alpha\beta\gamma\delta}=\frac{1}{n-1}(g_{\alpha\gamma}R_{\beta\delta}-g_{\alpha\delta}R_{\beta\gamma}) \tag{78}$$

The condition

$$R_{\alpha\beta\gamma\delta}+R_{\beta\alpha\gamma\delta}=0 \tag{79}$$

becomes

$$g_{\alpha\gamma}R_{\beta\delta}-g_{\alpha\delta}R_{\beta\gamma}+g_{\beta\gamma}R_{\alpha\delta}-g_{\beta\delta}R_{\alpha\gamma}=0 \tag{80}$$

and therefore

$$\sum_{\alpha,\gamma}g^{\alpha\gamma}(g_{\alpha\gamma}R_{\beta\delta}-g_{\alpha\delta}R_{\beta\gamma}+g_{\beta\gamma}R_{\alpha\delta}-g_{\beta\delta}R_{\alpha\gamma})=0 \tag{81}$$

that is

$$nR_{\beta\delta}-R_{\beta\delta}+R_{\beta\delta}-Rg_{\beta\delta}=0 \tag{82}$$

and we get

$$R_{\beta\delta}=\frac{R}{n}g_{\beta\delta} \tag{83}$$

where $R = \sum_{\alpha,\gamma} g^{\alpha\gamma} R_{\alpha\gamma}$ is the scalar curvature. By replacing (83) in (76) we obtain

$$R^{\alpha}_{\beta\gamma\delta} = \frac{R}{n(n-1)}(\delta^{\alpha}_{\gamma} g_{\beta\delta} - \delta^{\alpha}_{\delta} g_{\beta\gamma}). \tag{84}$$

We can conclude saying that a Riemannian manifold is locally projectively flat if and only if the sectional curvature is constant. Therefore while in classical general relativity the curvature tensor equal to zero means Minkowski spacetime, in projective general relativity curvature tensor equal to zero means De Sitter spacetime.

7 Conclusion

In this work we have analyzed the development of Projective Relativity by analyzing, particularly, its applications to cosmology. We have brought the results gotten by the author of this paper and by other authors in these last years. The cosmological applications of Projective Relativity seem to be interesting since they resolve many problems of the standard cosmology. In this scenario the space flatness is linked to the observer geometry and it is independent from the presence and distribution of matter-energy. This resolves the flatness problem without introducing inflationary hypotesis and the global spacetime structure is univocally individuated by the algebraic structure of the physical laws.

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